The relativistic statistical theory and Kaniadakis entropy: an approach through a molecular chaos hypothesis

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Abstract. We have investigated the proof of the H theorem within a manifestly covariant approach by considering the relativistic statistical theory developed in [G. Kaniadakis, Phys. Rev. E **66**, 056125 (2002); G. Kaniadakis, Phys. Rev. E **72**, 036108 (2005)]. As it happens in the nonrelativistic limit, the molecular chaos hypothesis is slightly extended within the Kaniadakis formalism. It is shown that the collisional equilibrium states (null entropy source term) are described by a κ power law generalization of the exponential Juttner distribution, e.g., $f(x, p) \propto (\sqrt{1 + \kappa^2\theta^2} + \kappa\theta)^{1/\kappa} \equiv \exp_\kappa \theta$, with $\theta = \alpha(x) + \beta_\mu p^\mu$, where $\alpha(x)$ is a scalar, β_μ is a four-vector, and p^μ is the four-momentum. As a simple example, we calculate the relativistic κ power law for a dilute charged gas under the action of an electromagnetic field $F^{\mu\nu}$. All standard results are readily recovered in the particular limit $\kappa \to 0$.

PACS. 05.90.+m Other topics in statistical physics, thermodynamics, and nonlinear dynamical systems – 05.20.-y Classical statistical mechanics

1 Introduction

The Boltzmann's famous H theorem, which guarantees positive-definite entropy production outside equilibrium, also describes the increase in the entropy of an ideal gas in an irreversible process, by considering the Boltzmann equation. Roughly speaking, this seminal theorem implies that in the equilibrium thermodynamic the distribution function of an ideal gas evolves irreversibly towards Maxwellian equilibrium distribution [1]. In the special relativistic domain, the very first derivation of this theorem was done by Marrot [2] and, in the local form, by Ehlers [3], Tauber and Weinberg [4] and Chernikov [5]. As well-known, the H theorem furnishes the Juttner distribution function for a relativistic gas in equilibrium, which contains the number density, the temperature, and the local four-momentum as free parameters [6].

Recently, this theorem has also been investigated in the context of a nonextensive statistic mechanics (NSM) [7]. In fact, the NSM has been proposed as a possible extension of the classical one, being a framework based on the deviations of Boltzmann-Gibbs-Shannon entropic measure [8]. It is worth mentioning that most of the experimental evidence supporting a NSM are related to the power-law distribution associated with the many-particle systems [9].

More recently, based on similar arguments, Abe [10,11] and Kaniadakis [12,13] have also proposed other entropic formulas. In this latter ones, the κ-entropy emerges in the context of the special relativity and in the so-called *kinetic interaction principle* (KIP). In particular, the relativistic H theorem in this approach has also been investigated through a self-consistent relativistic statistical theory [14] and through the framework of nonlinear kinetics [15].

Actually, this κ-framework leads to a class of one parameter deformed structures with interesting mathematical properties [16]. In particular, the so-called Lesche stability was checked in the κ-framework [17]. It was also shown that it is possible to obtain a consistent form for the entropy (linked with a two-parameter deformations of logarithm function), which generalizes the Tsallis, Abe and Kaniadakis logarithm behaviours [18]. In the experimental viewpoint, there exist some evidence related with the κ-statistics, namely, cosmic rays flux, rain events in meteorology [16], quark-gluon plasma [19], kinetic models describing a gas of interacting atoms and photons [20], fracture propagation phenomena [21], and income distribution [22], as well as construct financial models [23]. In the theoretical front, some studies on the canonical quantization of a classical system has also been investigated [24].

From the mathematical viewpoint, the κ-framework is based on κ-exponential and κ-logarithm functions, which

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is defined by

$$\exp_\kappa(f) = \left(\sqrt{1 + \kappa^2 f^2} + \kappa f \right)^{1/\kappa}, \qquad (1)$$

$$\ln_\kappa(f) = \left(f^\kappa - f^{-\kappa} \right) / 2\kappa. \qquad (2)$$

The κ-entropy associated with κ-framework is given by

$$S_\kappa(f) = -\int d^3p f \left[a_\kappa f^\kappa + a_{-\kappa} f^{-\kappa} + b_\kappa \right], \qquad (3)$$

which recovers standard Boltzmann-Gibbs entropy $S_{\kappa=0}(f) = -\int f \ln f d^3p$ in the limit $\kappa \to 0$, i.e., the $S_{\kappa=0}$ is obtained through the constraints on the constants a_κ and b_κ given by (see Refs. [12,13] for details)

$$\lim_{\kappa \to 0} [\kappa a_\kappa - \kappa a_{-\kappa}] = 1, \quad \lim_{\kappa \to 0} [b_\kappa + a_\kappa + a_{-\kappa}] = 0. \quad (4)$$

Hereafter the Boltzmann constant is set equal to unity for the sake of simplicity.

Previous works have already discussed some specific choices for the constants a_κ and b_κ, i.e., for the pair $[a_\kappa = 1/2\kappa, b_\kappa = 0]$, the Kaniadakis entropy reads [11,12]

$$S_\kappa = -\int d^3p f \ln_\kappa f = -\langle \ln_\kappa(f) \rangle. \qquad (5)$$

In particular, for entropy (5) the H theorem has been proved in reference [13]. Here, we consider the choice $[a_\kappa = 1/2\kappa(1 + \kappa), b_\kappa = -a_\kappa - a_{-\kappa}]$ with the κ-entropy given by [11]

$$S_\kappa = -\int d^3p \left(\frac{f^{1+\kappa}}{2\kappa(1 + \kappa)} - \frac{f^{1-\kappa}}{2\kappa(1 - \kappa)} + b_\kappa f \right). \quad (6)$$

In this paper, we intend to extend the nonrelativistic H theorem within the Kaniadakis framework to the special relativistic domain through a manifestly covariant approach. As we shall see, our approach does not consider the so-called deformed mathematics [13]. Rather, we show a proof for the H theorem based on similar arguments of references [25,26], e.g., a generalization of the molecular chaos hypothesis and of the four-entropy flux.

2 Classical H theorem

We first recall the basis for the proof of the standard H theorem within the special relativity. As well-known, the H theorem is also based on the molecular chaos hypothesis (Stosszahlansatz), i.e., the assumption that any two colliding particles are uncorrelated. This means that the two point correlation function of the colliding particles can be factorized

$$f(x, p, p_1) = f(x, p)f(x, p_1), \qquad (7)$$

or, equivalently,

$$\ln f(x, p, p_1) = \ln f(x, p) + \ln f(x, p_1), \qquad (8)$$

where p and p_1 are the four-momenta just before collision and the particles have four-momentum $p \equiv p^\mu = (E/c, \mathbf{p})$ in each point $x \equiv x^\mu = (ct, \mathbf{r})$ of the space-time, with their energy satisfying $E/c = \sqrt{\mathbf{p}^2 + m^2c^2}$. In order to complete the proof of the H theorem, we combine the Boltzmann equation with the four-divergence of the four-entropy flow, i.e.,

$$S^\mu = -c^2 \int \frac{d^3p}{E} p^\mu f \ln f. \qquad (9)$$

In this concern, it is possible to show that the relativistic Kaniadakis entropy is consistent with a slight departing from "Stosszahlansatz". Basically, this means the replacement of the logarithm functions appearing in (8) by κ-logarithmic (power laws) defined by equation (2). In reality, it is worth mentioning that the validity of the chaos molecular hypothesis still remains as a very controversial issue [27].

2.1 Generalized H theorem

In order to investigate H theorem in the context of the Kaniadakis statistics, we first consider a relativistic rarefied gas containing N point particles of mass m enclosed in a volume V, under the action of an external four-force field F^μ. Naturally, the states of the gas must be characterized by a Lorentz invariant one-particle distribution function $f(x, p)$, which the quantity $f(x, p)d^3x d^3p$ gives, at each time t, the number of particles in the volume element $d^3x d^3p$ around the particles space-time position x and momentum $\mathbf{p}$. By considering that every influence of a power law statistic must happen within the collisional term of Boltzmann equation (see also [25,26]), we assume that the temporal evolution of the relativistic distribution function $f(x, p)$ is given by the following κ-transport equation

$$p^\mu \partial_\mu f + mF^\mu \frac{\partial f}{\partial p^\mu} = C_\kappa(f), \qquad (10)$$

where $\mu = 0, 1, 2, 3$ and $\partial_\mu = (c^{-1}\partial_t, \nabla)$ indicates differentiation with respect to time-space coordinates and C_κ denotes the relativistic κ-collisional term. Following the same physical arguments concerning the collisional term from approach of references [25,26], we have that $C_\kappa(f)$ has the general form

$$C_\kappa(f) = \frac{c}{2} \int F\sigma R_\kappa(f, f') \frac{d^3p_1}{E_1} d\Omega, \qquad (11)$$

where $d\Omega$ is an element of the solid angle, the scalar F is the invariant flux, which is equal to $F = \sqrt{(p_\mu p_1^\mu)^2 - m^4c^4}$, and σ is the differential cross section of the collision $p + p_1 \to p' + p_1'$; see reference [28] for details. All quantities are defined in the center-of-mass system of the colliding particles. Next, we observe that C_κ must be consistent with the energy, momentum, and the particle number conservation laws, and its specific structure must be such that the standard result is recovered in the limit $\kappa \to 0$.

Here, the κ-generalized form of molecular chaos hypothesis is also a difference of two correlation functions

$$R_\kappa(f, f') = \exp_\kappa\left(\ln_\kappa f' + \ln_\kappa f'_1\right) - \exp_\kappa\left(\ln_\kappa f + \ln_\kappa f_1\right), \quad (12)$$

where primes refer to the distribution function after collision, and $\exp_\kappa(f)$, $\ln_\kappa(f)$, are defined by equations (1) and (2). Note that for $\kappa = 0$, the above expression reduces to $R_0 = f'f'_1 - ff_1$, which is exactly the standard molecular chaos hypothesis. In the present framework, the κ four-entropy flux reads

$$S^\mu_\kappa = -c^2 \int \frac{d^3p}{E}p^\mu\left(\frac{f^{1+\kappa}}{2\kappa(1+\kappa)} - \frac{f^{1-\kappa}}{2\kappa(1-\kappa)} + b_\kappa f\right), \quad (13)$$

where for $\mu = 0$, the quantity $c^{-1}S^0_\kappa$ stands for the local Kaniadakis' entropy density, as given by (6). Indeed, by taking the four-divergence of S^μ_κ, i.e.,

$$\partial_\mu S^\mu_\kappa \equiv \tau_\kappa = -c^2 \int \frac{d^3p}{E}p^\mu\partial_\mu f(\ln_\kappa f + b_\kappa), \quad (14)$$

and combining with κ relativistic Boltzmann equation (10), we obtain the source term

$$\tau_\kappa = -\frac{c^3}{2} \int \frac{d^3p}{E}\frac{d^3p_1}{E_1}d\Omega F\sigma R_\kappa(\ln_\kappa f + b_\kappa). \quad (15)$$

Next, τ_κ can be written in a more symmetrical form by using some elementary symmetry operations, which also take into account the inverse collisions. Let us notice that by interchanging p and p_1 the value of the integral above is not modified. This happens because the scattering cross section and the magnitude of the flux are invariants [28]. The value of τ_κ is not altered if we integrate with respect to the variables p' and p'_1. Although changing the sign of R_κ in this step (inverse collision), the quantity $d^3pd^3p_1/p^0p^0_1$ is also a collisional invariant [28]. As one may check, such considerations imply that the κ-entropy source term can be written as

$$\tau_\kappa(x) = \frac{c^3}{8} \int \frac{d^3p}{E}\frac{d^3p_1}{E_1}d\Omega F\sigma\big(\ln_\kappa f'_1 + \ln_\kappa f'$$
$$- \ln_\kappa f_1 - \ln_\kappa f\big)\big[\exp_\kappa\left(\ln_\kappa f' + \ln_\kappa f'_1\right)$$
$$- \exp_\kappa\left(\ln_\kappa f + \ln_\kappa f_1\right)\big]. \quad (16)$$

As is well-known, the irreversibility thermodynamics emerging from molecular collisions is quite obtained if $\tau_\kappa(x)$ is positive definite. This condition is guaranteed only when the integrand in (16) is not negative. Indeed, by introducing the auxiliary functions, namely $z = \exp_\kappa(\ln_\kappa f + \ln_\kappa f_1)$ and $y(z) = \ln_\kappa f + \ln_\kappa f_1$. The differences $z' - z$ and $y' - y$ can be positive or negative and these differences have the same sign if the functions y is an increasing function. However, $\ln_\kappa f$ is an increasing function and then product $(y' - y)(z' - z)$ is always positive for any pair of distributions (f, f_1) and (f', f'_1). Therefore, the positiveness of τ_κ must be established [28,29].

For the sake of completeness, let us derive the version of the Juttner distribution within the κ-statistic. Such an expression is the relativistic version of the κ-distribution [16], and must be obtained as a natural consequence of the relativistic H theorem. At this point, it is interesting to emphasize that such a distribution already has been introduced by Kaniadakis through a variational problem in a selfconsistent approach; see reference [14] for details. The H theorem states that $\tau_\kappa = 0$ is a necessary and sufficient condition for equilibrium. Since the integrand of (16) cannot be negative, this occurs if and only if

$$\ln_\kappa f' + \ln_\kappa f'_1 = \ln_\kappa f + \ln_\kappa f_1, \quad (17)$$

where the four-momenta are connected through a conservation law

$$p^\mu + p^\mu_1 = p'^\mu + p'^\mu_1,$$

which is valid for any binary collision. Therefore, the above sum of κ-logarithms remains constant during a collision: it is a summational invariant. In the relativistic case, the most general collisional invariant is a linear combination of a constant plus the four-momentum p^μ; see reference [28]. Consequently, we must have

$$\ln_\kappa f(x, p) = \theta = \alpha(x) + \beta_\mu p^\mu, \quad (18)$$

where $\alpha(x)$ is a scalar, β_μ a four-vector, and p^μ is the four-momentum. By using that $\exp_\kappa(\ln_\kappa f) = f$, we may rewrite (18) as

$$f(x, p) = \exp_\kappa \theta = \left(\sqrt{1 + \kappa^2\theta^2} + \kappa\theta\right)^{1/\kappa}, \quad (19)$$

with arbitrary space and time-dependent parameters $\alpha(x)$ and $\beta_\mu(x)$. Some considerations on the function $f(x, p)$ are given as follows. First, this is the most general expression which leads to a vanishing collision term and entropy production, and reduces to Juttner distribution in the limit $\kappa \to 0$. However, it is not true in general that $f(x, p)$ is a solution of the transport equation. This happens only if $f(x, p)$ also makes the left-hand-side of the transport equation (10) to be identically null. Nevertheless, since (19) is a power law, the transport equation implies that the parameters $\alpha(x)$ and $\beta_\mu(x)$ must only satisfy the constraint equation

$$p^\mu\partial_\mu\alpha(x) + p^\mu p^\nu\partial_\mu\beta_\nu(x) + m\beta_\mu(x)F^\mu(x, p) = 0. \quad (20)$$

Second, the above expression is also the relativistic version of the Kaniadakis distribution [12]. Here, this was obtained through the different approach from the one used in references [13,14].

As an example, let us now consider a relativistic gas under the action of the Lorentz 4-force

$$F^\mu(x, p) = -(q/mc)F^{\mu\nu}(x)p_\nu,$$

where q is the charge of the particles and $F^{\mu\nu}$ is the Maxwell electromagnetic tensor. Following standard lines, it is easy to show that the local equilibrium function in the presence of an external electromagnetic field reads

$$f(x, p) = \exp_\kappa\left[\frac{\mu - [p^\mu + c^{-1}qA^\mu(x)]U_\mu}{T}\right], \quad (21)$$

where U_μ is the mean four-velocity of the gas, $T(x)$ is the temperature field, μ is the Gibbs function per particles, and $A^\mu(x)$ the four potential. As physically expected, note also that the above expression reduces, in the limit $\kappa \to 0$, to the well known expression [28,30]

$$f(x,p) = \exp\left(\frac{\mu - [p^\mu + c^{-1}qA^\mu(x)]U_\mu}{T}\right). \qquad (22)$$

3 Conclusions

In references [25,26] we have discussed the H theorem in the context from the Kaniadakis and Tsallis statistics within the nonrelativistic and relativistic domain. Based on the generalization of the chaos molecular hypothesis and the entropic measure, it was shown the proof of the H theorem in both domain. In this paper, by considering the same arguments on the chaos molecular and entropy, and regardless of the deformed mathematics introduced in reference [13], we have studied a κ-generalization of the relativistic Boltzmann's kinetic equation along the lines defined by the Kaniadakis statistics. In reality, since the basic results were obtained through a manifestly covariant way, their generalization to the general relativistic domain may be readly derived. Finally, we also emphasize that our proof is consistent with the standard laws describing the microscopic dynamics, and reduce to the familiar Boltzmann proof in the limit $\kappa \to 0$.

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